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TAYLOR DIFFUSION IN A FALLING FILM OF A NON-NEWTONIAN LIQUID

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Abstract—Diffusion of a solute in a laminar film of an Eyring liquid down a vertical plate is considered. The Taylor diffusion coefficient for this film increases with the model parameter.

NOMENCLATURE

- a, radius of the pipe;
- A, B, Eyring model parameter;
- C, concentration of the solute;
- D, molecular diffusion coefficient;
- D^*, Taylor diffusion coefficient;
- d, thickness of the film;
- E, constant defined by (8);
- g, acceleration due to gravity;
- G, dimensionless Eyring model parameter defined by equation (6);
- t, time;
- $u(y)$, velocity distribution in the flow;
- U, average velocity.

Greek symbols

- ξ, non-dimensional axial co-ordinate axis $(x-Ut)/d$;
- η, non-dimensional transverse co-ordinate axis y/d;
- τ, non-dimensional time Dt/d^2;
- ρ, density;
- μ, coefficient of viscosity;
- χ, dimensionless velocity defined by (6);
- λ_1, constant defined by (6).

INTRODUCTION

THE DISPERSION of a solute in a viscous liquid flowing in a circular pipe under laminar conditions was investigated by Taylor [1, 2]. It turns out that relative to a plane moving with the mean speed of the flow, the solute is dispersed (subject to certain limitations) with an apparent diffusion coefficient $a^2v_x^2/48D$, where a, v_x and D are the radius of the pipe, the average velocity and the molecular diffusion coefficient respectively. Aris [3] extended Taylor's analysis and presented a more detailed treatment of the flow after removing the restrictions imposed by Taylor. Recently Prenosil [4] studied the dispersion of a solute in a laminar falling film of a viscous liquid. This analysis has some bearing on the treatment of the residence time distribution of such a film. It may also be noted that the phenomenon of dispersion in liquid films is encountered in such processes as absorption, humidification, evaporation and extraction.

The flow characteristics of thin films of non-Newtonian power law fluids down a vertical plate were examined by Sylvester *et al.* [5]. However, the phenomenon of diffusion in a falling film of non-Newtonian liquids has not received much attention, despite the fact that such liquids are of importance in chemical industries. The object of the present paper is to study the dispersion of a solute in a laminar falling film of a non-Newtonian liquid obeying the Eyring model [6]. The chief merits of this two-parameter model are that it is derivable from the kinetic theory of liquids and that it predicts pseudoplastic behaviour at finite values of the shear stress. Further, the constitutive equation for an Eyring liquid reverts to that for an ordinary Newtonian liquid when the shear stress approaches zero.

DIFFUSION IN AN EYRING LIQUID

Consider the laminar unidirectional flow of a film of an Eyring liquid down a vertical plate with X-axis along the free surface of the film downwards and with Y-axis normal and directed towards the plate. If $u(y)$ and τ_{xy} denote the velocity and shear stress at a point, then according to the Eyring model [6],

$$-\frac{\mathrm{d}u}{\mathrm{d}y} = B\sinh\left(\frac{\tau_{xy}}{A}\right), \tag{1}$$

where A and B are the model parameters. It is easy to see that (1) reverts to the constitutive equation for a Newtonian liquid with coefficient of viscosity μ as $A \to \infty$ and $B \to \infty$ such that

$$\mu = \lim_{\substack{A\to\infty \\ B\to\infty}} (A/B).$$

The momentum equation for the liquid film of thickness d is

$$-\frac{\mathrm{d}\tau_{xy}}{\mathrm{d}y} + \rho g = 0, \tag{2}$$

where ρ and g denote the density of the liquid and the acceleration due to gravity. The solution of (1) and (2) satisfying the zero shear stress condition $\mathrm{d}u/\mathrm{d}y = 0$ at the free surface $y = 0$ and the no-slip condition $u = 0$ at the plate $y = d$ is

$$u = \frac{AB}{\rho g}\left[\cosh\left(\frac{\rho g d}{A}\right) - \cosh\left(\frac{\rho g y}{A}\right)\right]. \tag{3}$$

Introducing the average velocity U given by

$$U = \frac{1}{d}\int_0^d u(y)\,\mathrm{d}y, \tag{4}$$

we may put $u(y)$ in (3) as

$$u = U[1+\chi(\eta)], \tag{5}$$

where

$$\chi(\eta) = \frac{1}{\lambda_1}\left[1-\frac{G\cosh G\eta}{\sinh G}\right],\quad G = \frac{\rho g d}{A},\quad \eta = y/d,$$
$$\lambda_1 = G\cosh G - 1,\quad U = \frac{Bd}{G}\left[\cosh G - \frac{\sinh G}{G}\right]. \tag{6}$$

The concentration C of the solute diffusing in the film satisfies

$$\frac{\partial C}{\partial t} = D\nabla^2 C - U(1+\chi)\frac{\partial C}{\partial x}, \tag{7}$$

where D is the molecular diffusivity (assumed constant).

Considering a frame of reference with origin moving with the mean velocity of the film we introduce the following dimensionless quantities

$$\xi = \frac{x-Ut}{d},\quad \tau = \frac{Dt}{d^2},\quad E = \frac{Ud}{D}. \tag{8}$$

Use of (8) in (7) gives

$$\frac{\partial C}{\partial \tau} = \frac{\partial^2 C}{\partial \xi^2} - E\chi\frac{\partial C}{\partial \xi} + \frac{\partial^2 C}{\partial \eta^2}, \tag{9}$$

with the initial and the boundary conditions

$$C(\xi,\eta,0) = C_0(\xi,\eta), \tag{10}$$

$$\frac{\partial C}{\partial \eta} = 0 \quad \text{at} \quad \eta = 0 \text{ and } \eta = 1. \tag{11}$$

Equation (11) represents zero mass flux at the free surface and the plate.

Let us introduce $C_p(\eta,\tau)$ the pth moment of the distribution of the solute in the film through η at time t:

$$C_p(\eta,\tau) = \int_{-\infty}^{\infty} \xi^p C(\xi,\eta,\tau)\,\mathrm{d}\xi. \tag{12}$$

We also introduce the pth moment $m_p(\tau)$ in the film given by

$$m_p(\tau) = \int_0^1 C_p(\eta,\tau)\,\mathrm{d}\eta. \tag{13}$$

Following the method similar to Aris, we have from (9) and (12),

$$\frac{\partial C_p}{\partial \tau} = \frac{\partial^2 C_p}{\partial \eta^2} + p(p-1)C_{p-2} + Ep\chi C_{p-1}, \tag{14}$$

subject to the initial and boundary conditions

$$C_p(\eta,0) = C_{p_0}(\eta), \tag{15}$$

$$\frac{\partial C_p}{\partial \eta} = 0 \quad \text{at} \quad \eta = 0 \text{ and } 1. \tag{16}$$

Averaging (14) over the film thickness, and using (13) and (16), we get

$$\frac{\partial m_p}{\partial \tau} = p(p-1)m_{p-2} + Ep\int_0^1 \chi C_{p-1}\,\mathrm{d}\eta, \tag{17}$$

with condition (15) transformed to

$$m_p(0) = m_{p_0}. \tag{18}$$

Solving (14) and (17) successively and using the initial and boundary conditions (15), (16) and (18), we have

$$m_0 = \text{constant}, \tag{19}$$

$$C_0(\eta,\tau) = 1 + \sum_1^{\infty} A_n' \cos n\pi\eta \cdot \mathrm{e}^{-n^2\pi^2\tau}, \tag{20}$$

$$m_1(\tau) = -\frac{EG^2}{\lambda_1}\sum_1^{\infty}\frac{A_n'\cos n\pi}{(G^2+n^2\pi^2)n^2\pi^2}\cdot(1-\mathrm{e}^{-n^2\pi^2\tau}), \tag{21}$$

where the origin is selected in the initial plane of the centre of gravity of the solute so that $m_1(0) = 0$. Further, the constant in (19) may be taken unity without loss of generality so that $m_0 = 1$ and

$$A_n' = 2\int_0^1 C_\infty(\eta)\cos n\pi\eta\,\mathrm{d}\eta,\quad n = 1,2,\ldots. \tag{22}$$

Equation (19) merely expresses the fact that the total quantity of the solute remains constant. Further, as $\tau \to \infty$, equation (21) gives

$$m_1(\infty) = -\frac{EG^2}{\lambda_1}\sum_1^{\infty}\frac{A_n'\cos n\pi}{(G^2+n^2\pi^2)n^2\pi^2}, \tag{23}$$

which is the ultimate position to which the centre of gravity of the solute moves. It may be noted from (21) that since $\mathrm{d}m_1/\mathrm{d}\tau \to 0$ as $\tau \to \infty$, the centre of gravity of the solute ultimately moves with the mean speed of the flow.

Now equation (14) gives for $p = 1$,

$$\frac{\partial C_1}{\partial \tau} = \frac{\partial^2 C_1}{\partial \eta^2} + E\chi C_0. \tag{24}$$

Substituting (20) in (24) and solving in terms of the complementary function and the particular integral, we have after using (15) and (16):

$$C_1(\eta,\tau) = -\frac{E}{\lambda_1}\left[\frac{\eta^2}{2} - \frac{\cosh G\eta}{G\sinh G}\right] + \tfrac{1}{2}B_0'$$
$$+ \sum_1^{\infty} B_n'\cos n\pi\eta\cdot\mathrm{e}^{-n^2\pi^2\tau} + E\sum_1^{\infty}\phi_n(\eta)\cdot\mathrm{e}^{-n^2\pi^2\tau}, \tag{25}$$

where $\phi_n(\eta)$ satisfies

$$\frac{\mathrm{d}^2\phi_n}{\mathrm{d}\eta^2} + n^2\pi^2\phi_n = -\chi A_n'\cos n\pi\eta, \tag{26}$$

subject to

$$\frac{\mathrm{d}\phi_n}{\mathrm{d}\eta} = 0 \quad \text{at} \quad \eta = 0 \text{ and } 1. \tag{27}$$

In (25), the constants B_n' are given by

$$B_n' = 2\int_0^1 C_{10}(\eta)\cos n\pi\eta\,\mathrm{d}\eta,\quad n = 0,1,2,\ldots. \tag{28}$$

To evaluate B_0', we substitute $C_1(\eta,\tau)$ from (25) in (13) with $p = 1$ and make use of the condition $m_1(0) = 0$. This gives

$$\tfrac{1}{2}B_0' = \frac{E}{\lambda_1}\left(\tfrac{1}{6} - \frac{1}{G^2}\right) - E\sum_1^{\infty}\bar{\phi}_n. \tag{29}$$

Substituting (29) in (25) and taking the limit $\tau \to \infty$, we find

$$C_1(\eta, \infty) \sim m_{1\infty} + \frac{E}{\lambda_1}\left[\frac{1}{6} - \frac{1}{G^2} - \frac{\eta^2}{2} + \frac{\cosh G\eta}{G \sinh G}\right]. \quad (30)$$

Now putting $p = 2$ in (17) and using $m_0 = 1$ and the expression for $C_1(\eta, \tau)$, we get after neglecting the terms which tend to zero as $\tau \to \infty$:

$$\frac{dm_2}{d\tau} = 2 + 2E\int_0^1 \chi C_1 \, d\eta$$

$$\simeq 2 + \frac{2E^2}{\lambda_1^2 \sinh G}\left[\left(\frac{1}{3} + \frac{2}{G^2}\right)\sinh G - \frac{3}{2G}\cosh G - \frac{1}{2\sinh G}\right]. \quad (31)$$

Now the rate of change of variance V_{ar} of the distribution of the solute about the moving origin is proportional to

$$\lim_{t\to\infty} (dm_2/dt).$$

Thus using $\tau = tD/d^2$ and (13), we have

$$\frac{dV_{ar}}{dt} \propto D + D^*, \quad (32)$$

where

$$D^* = \frac{U^2 d^2}{D\lambda_1^2 \sinh G}\left[\left(\frac{1}{3} + \frac{2}{G^2}\right)\sinh G - \frac{3\cosh G}{2G} - \frac{1}{2\sinh G}\right]. \quad (33)$$

Substituting the value of U given by (6) in (33), we may write D^* as $(B^2 d^4/D) \cdot F_1(G)$ where

$$F_1(G) = \frac{1}{G^4}\left[\left(\frac{1}{3} + \frac{2}{G^2}\right)\sinh^2 G - \frac{3}{4G}\sinh 2G - \frac{1}{2}\right]. \quad (34)$$

We have computed $F_1(G)$ for various values of G in Table 1 shown below.

Table 1.

G	$F_1(G)$
1.0	0.24163×10^{-2}
2.0	0.14253×10^{-1}
3.0	0.59581×10^{-1}
4.0	0.23974
5.0	0.99740
10.0	0.24663×10^{4}
15.0	0.12783×10^{8}

We thus see from (32) that the total diffusion coefficient may be looked upon as the sum of the molecular diffusion coefficient D and the effective Taylor diffusion coefficient D^*. Further, for a fixed value of B, D^* increases rapidly with increase in the model parameter G. It may be shown from (34) that

$$\lim_{G\to 0} F_1(G) = 0$$

and this may lead one to believe that the Taylor diffusion coefficient for the falling film given by (33) tends to zero in the viscous limit $A \to \infty$, i.e. $G \to 0$. This apparently contradicts the result of Prenosil. However, this paradox may be resolved by recalling the fact that the correct viscous limit is obtained by taking $A \to \infty$ (i.e. $G \to 0$) and $B \to \infty$ subject to the condition that $\lim_{\substack{G\to 0\\ B\to\infty}} [B^2 F_1(G)]$ remains finite.

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DIFFUSION DE TAYLOR DANS UN LIQUIDE NON NEWTONIEN TOMBANT EN FILM

Résumé—On considère la diffusion d'un soluté dans un film laminaire de liquide d'Eyring qui tombe le long d'une plaque verticale. Le coefficient de diffusion selon Taylor croît avec le paramètre de modèle.

TAYLOR-DIFFUSION IN EINEM RIESELFILM EINER NICHT-NEWTONSCHEN FLÜSSIGKEIT

Zusammenfassung—Für einen in Lösung gehenden Stoff wird die Diffusion in einen laminaren Film einer Eyring-Flüssigkeit an einer vertikalen Platte betrachtet. Der Taylor-Diffusionskoeffizient für diesen Film nimmt mit den Modell-Parametern zu.

ТЕЙЛОРОВСКАЯ ДИФФУЗИЯ СТЕКАЮЩЕЙ ПЛЕНКИ НЕНЬЮТОНОВСКОЙ ЖИДКОСТИ

Аннотация — Рассматривается диффузия растворенного вещества в ламинарной пленке жидкости Эйринга, стекающей с вертикальной пластины. С ростом безразмерного индекса реологической модели коэффициент тейлоровской диффузии увеличивается для данной пленки.